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Taylor instability of a non-uniform free-surface flow

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The evolution of a small disturbance in a three-dimensional steady free-surface flow is investigated. The radius of curvature of the free surface and the length scale characterizing the non-uniformity of the velocity are assumed to be of the same order of magnitude. It is shown that the local rate of growth of the amplitude of the disturbance depends on both the normal pressure gradient (as in the case of Taylor instability) and the rate of strain on the free surface. Application of the theory to rising gaseous bubbles and gravity water waves is discussed.

1. Introduction

In a previous study (Dagan & Tulin 1972) we have suggested a mechanism of inception of breaking of a two-dimensional free-surface gravity flow in front of a blunt body (figure 1): as the Froude number grows, the centrifugal acceleration V'^2/R' is assumed to offset the normal component of the gravitational acceleration $g' \cos\theta$ at a certain point A of convexity of the free surface (V' is the velocity modulus, R' is the radius of curvature of the free surface and θ the angle between the normal and the vertical). This balance on the local scale was supposed to be tantamount to the Taylor criterion for instability of a free surface accelerated from below (Taylor 1950) and adopted, thenceforth, as a criterion for stability of the flow of figure 1. The Taylor criterion is valid, however, in the case of a plane, horizontal unperturbed free surface and a uniform flow parallel to the free surface. Its direct extension to a steady non-uniform flow, as suggested in our previous work, may be valid under a few restrictive conditions, which are now examined. For this purpose let us define two characteristic local scales in the neighbourhood of a point on the free surface: $l' = U'/(\partial V'/\partial s')$, characterizing the flow non-uniformity (U' is a characteristic velocity and s' is a co-ordinate along the free surface), and R', the radius of curvature. Testing the stability of the flow to a small disturbance of wavelength λ' may lead to the aforementioned simplified criterion if (i) $V'^2/g'R' = O(1)$, (ii) $\lambda' \ll R'$ and (iii) $R' \ll l'$. These conditions imply that on the local λ' scale the flow is almost uniform and the disturbance travels many wavelengths in a region subjected to a constant normal acceleration.

The two lengths l' and R' are not independent, however, and condition (iii) is generally not satisfied. Let $F = V'/(gh')^{\frac{1}{2}}$ be the Froude number characterizing the flow of figure 1. It can be shown that $l' = O(h')$ and $R = O(gh'^2/V'^2)$, so that, at small F, $l' \ll R'$ while at $F = O(1)$, l' is probably of the order of R', but not much larger as implied by (iii).

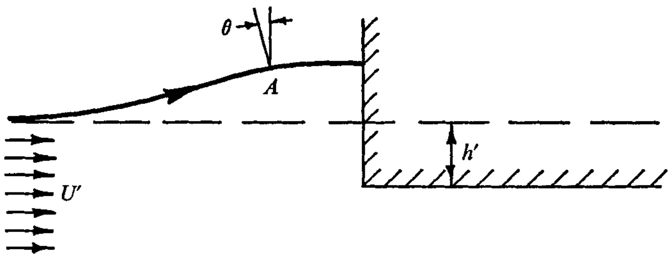

FIGURE 1. Steady free-surface gravity flow before a blunt body.

The aim of the present work is to generalize the Taylor instability criterion for a small disturbance to a steady non-uniform flow in the general case when l' and R' are arbitrary. It will turn out that our generalized criterion degenerates into the Taylor criterion under the condition $l' \gg R'$.

The problem is also related to previous studies of the evolution of short gravity waves on steady non-uniform currents (Longuet-Higgins & Stewart 1960, 1961; Whitham 1962). In these studies the non-uniform part of the basic flow is assumed from the outset to be small and of negligible radius of curvature, i.e. $R' \gg l'$. Here we do not impose any restriction upon the basic flow and we leave the relationship between R' and l' unspecified. The stability of a few free-surface, gravity-free, two-dimensional flows is also reviewed by Wu (1968), who neglects the effect of gravity and surface tension upon the disturbance.

The derivation of the generalized stability criterion may be helpful in the understanding of the important phenomenon of the breaking of the free surface of a gravity flow.

2. The linearized equations

We consider a steady inviscid three-dimensional flow with a free surface (figure 2). The flow is related to local Cartesian co-ordinates (τ', σ', ν'), where τ' and σ' lie in a plane tangential to the free surface and ν' is normal to it and directed outwards from the fluid. The variables are first made dimensionless by referring them to g' (the acceleration due to gravity) and L' (a characteristic length), i.e.

$$\left.\begin{aligned} &\Phi = \Phi'/L'(g'L')/^{\frac{1}{2}}, \quad \tau = \tau'/L', \quad \sigma = \sigma'/L', \quad \nu = \nu'/L', \\ &V = V'/(g'L')^{\frac{1}{2}}, \quad t' = t'(g'/L')^{\frac{1}{2}}, \ldots, \end{aligned}\right\} \tag{1}$$

where $\Phi'(\sigma', \tau', \nu)$ is the velocity potential of the basic steady flow and t' is the time.

We consider now a perturbation of the basic flow characterized by a small parameter ϵ, the ratio between the amplitude of the perturbation and L', and expand as follows:

$$\phi(\tau, \sigma, \nu, t) = \Phi(\tau, \sigma, \nu) + \epsilon\phi_1(\tau, \sigma, \nu, t) + \ldots, \tag{2}$$

$$\eta(\tau, \sigma, t) = N(\tau, \sigma) + \epsilon\eta_1(\tau, \sigma, t) + \ldots, \tag{3}$$

$$z = Z + \epsilon z_1 + \ldots, \tag{4}$$

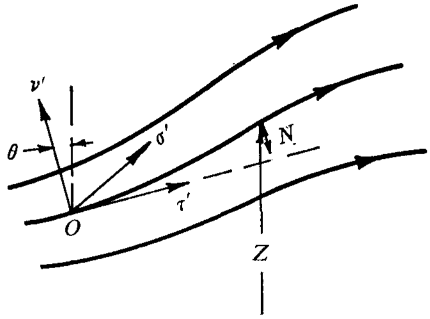

FIGURE 2. Co-ordinate system relative to the free surface of the basic flow.

where ϕ, η and z are the potential, the free-surface equation and the free-surface elevation, respectively, while Φ, N and Z and ϕ_1, η_1 and z_1 are the same quantities for the basic steady flow and the first-order perturbation, respectively.

Substitution of (2)–(4) in Laplace's equation, and the Bernoulli and kinematical free-surface conditions, and separation of terms $O(1)$ and $O(\epsilon)$ lead to the following sets of equations:

$$\nabla^2\Phi = 0 \quad (\nu \leqslant N), \tag{5}$$

$$\tfrac{1}{2}(\nabla\Phi)^2 + Z = \text{constant} \quad (\nu = N), \tag{6}$$

$$\frac{\partial\Phi}{\partial\tau}\frac{\partial N}{\partial\tau} + \frac{\partial\Phi}{\partial\sigma}\frac{\partial N}{\partial\sigma} - \frac{\partial\Phi}{\partial\nu} = 0 \quad (\nu = N); \tag{7}$$

$$\nabla^2\phi_1 = 0 \quad (\nu \leqslant N), \tag{8}$$

$$\frac{\partial\phi_1}{\partial t} + \nabla\Phi\,.\,\nabla\phi_1 - \frac{\partial P}{\partial\nu}\eta_1 = T\left(\frac{\partial^2\eta_1}{\partial\tau_2} + \frac{\partial^2\eta_1}{\partial\sigma^2}\right) \quad (\nu = N), \tag{9}$$

$$\left.\begin{aligned} &\frac{\partial\eta_1}{\partial t} + \frac{\partial\Phi}{\partial\tau}\frac{\partial\eta_1}{\partial\tau} + \frac{\partial\Phi}{\partial\sigma}\frac{\partial\eta_1}{\partial\sigma} + \frac{\partial N}{\partial\tau}\frac{\partial\phi_1}{\partial\tau} + \frac{\partial N}{\partial\sigma}\frac{\partial\phi_1}{\partial\sigma} \\ &\quad - \frac{\partial\phi_1}{\partial\nu} + \eta_1\frac{\partial}{\partial\nu}\left(\frac{\partial N}{\partial\tau}\frac{\partial\Phi}{\partial\tau} + \frac{\partial N}{\partial\sigma}\frac{\partial\Phi}{\partial\sigma} - \frac{\partial\Phi}{\partial\nu}\right) = 0 \end{aligned}\right\} \quad (\nu = N). \tag{10}$$

Equations (5)–(7) are the exact nonlinear equations of the basic steady flow with the effect of the surface tension neglected.

The term $\partial P/\partial\nu$ in (9) is the ν component of the pressure gradient of the basic flow at the free surface. It can be shown that at $\tau = 0$

$$\frac{\partial P}{\partial\nu} = -\nabla\Phi\,.\,\frac{\partial}{\partial\nu}\nabla\Phi - \cos\theta = -\left(\frac{V^2}{R} + \cos\theta\right). \tag{11}$$

The term on the right-hand side of (9) represents the effect of surface tension, T being equal to $T'/\rho' g' L'^2$, where T' is the coefficient of surface tension and ρ' the density.

If the basic flow and the initial value of ϕ_1 and η_1 are given, (8)–(10) permit us in principle to determine ϕ_1 and η_1 at any subsequent t. This is, however, a very difficult task because there are very few closed analytical solutions for steady free-surface flows and because these equations, although linear, have variable coefficients. We consider now a simplified analysis based on a local expansion of Φ and N

3. Local analysis

We assume that the wavenumber $k = 2\pi/\lambda$ characterizing the disturbance is large ($k \gg 1$). Moreover, we consider the evolution of the disturbance in the vicinity of the origin (figure 2), i.e. over a distance $r = (\sigma^2+\tau^2)^{\frac{1}{2}} = O(1)$, and we assume that $kr = O(1)$ and consequently, $kr^2 = o(1)$.

Under these conditions it is useful to expand Φ and N in a Taylor series in the neighbourhood of O for a regular point of the free surface, as follows:

$$N = \tfrac{1}{2}(a\tau^2+2b\tau\sigma+c\sigma^2)+\ldots, \tag{12}$$

$$\Phi = A\tau+B\sigma+C\nu+\tfrac{1}{2}(D\tau^2+E\sigma^2+G\nu^2+2M\tau\sigma+2Q\sigma\nu+2K\tau\nu)+\ldots, \tag{13}$$

where $a, b, \ldots, A, B, \ldots$, are constants. Substitution of (12) and (13) in (5)–(7) yields

$$A = \frac{\partial\Phi}{\partial\tau}, \quad B = \frac{\partial\Phi}{\partial\sigma}, \quad C = 0, \quad G = -(D+E) = \frac{\partial^2\Phi}{\partial\nu^2} \quad (\tau = \sigma = \nu = 0). \tag{14}$$

G has the following interpretation, independent of the co-ordinate system:

$$G = \frac{\partial^2\Phi}{\partial\nu^2} = -\frac{1}{\delta S}\frac{D(\delta S)}{Dt} \quad (\nu = \tau = \sigma = 0), \tag{15}$$

where δS is an element of area in the free surface of the basic flow and D/Dt is a material derivative. In the case of a two-dimensional flow (no dependence on σ) G may also be related to the acceleration or the slope as follows:

$$G = -\frac{\partial V}{\partial\tau} = \frac{1}{V}\frac{\partial Z}{\partial\tau} \quad (\nu = \tau = 0), \tag{16}$$

where $V = \partial\Phi/\partial\tau$. The flow will be called a stretching flow at a point where $G < 0$, while the motion will be called a contractive motion for $G > 0$. Equation (15) results from the expression for the material derivative of an element of area (Batchelor 1967, p. 132).

Substitution of N from (12) and Φ from (13) and (14) in (9) and (10) yields, after neglecting terms quadratic in τ and σ compared with linear terms, the following linear system satisfied by ϕ_1 and η_1:

$$\frac{\partial\phi_1}{\partial t}+A\frac{\partial\phi_1}{\partial\sigma}+B\frac{\partial\phi_1}{\partial\sigma}-\frac{\partial P}{\partial\nu}\eta_1 = T\left(\frac{\partial^2\eta_1}{\partial\tau^2}+\frac{\partial^2\eta_1}{\partial\sigma^2}\right) \quad (\nu = 0), \tag{17}$$

$$\frac{\partial\eta_1}{\partial t}+A\frac{\partial\eta_1}{\partial\tau}+B\frac{\partial\eta_1}{\partial\sigma}-\frac{\partial\phi_1}{\partial\tau}-G\eta_1 = 0 \quad (\nu = 0), \tag{18}$$

where $\partial P/\partial\nu$ is the normal pressure gradient of the basic flow at the origin.

We solve (8), (17) and (18) for the simple case of a wave of constant wavenumber k travelling at an angle β with respect to the τ axis, i.e.

$$\eta_1(\tau,\sigma,t) = \alpha(t)\exp\left[ik(\tau\cos\beta+\sigma\sin\beta)\right], \tag{19}$$

$$\phi_1(\tau,\sigma,\nu,t) = \gamma(t)\exp\left[ik(\tau\cos\beta+\sigma\sin\beta-i\nu)\right], \tag{20}$$

which yields, after substitution and solving the resulting ordinary differential equations,

$$\eta_1(\tau,\sigma,t) = \eta_1(0)\exp\{ik[\tau\cos\beta+\sigma\sin\beta-(\partial\Phi/\partial\beta)\,t]\}\,e^{\mu t}, \tag{21}$$

$$\mu = \tfrac{1}{2}\{G+[G^2+4k(\partial P/\partial\nu-k^2T)]^{\frac{1}{2}}\}. \tag{22}$$

The first exponential function on the right-hand side of (21) represents a wave of constant amplitude propagating in the β direction with the local velocity $\partial\Phi/\partial\beta\,(\nu=\tau=\sigma=0)$. μ is the rate of change of the amplitude and its derivation is the main quantitative result of the present analysis.

4. Local growth or decay of a small disturbance

First we shall show that μ, given by (22), degenerates into simple known expressions in a few extreme cases.

(i) If $G=0$, i.e. the flow is locally uniform, (22) yields

$$\mu = \mu_T = [k(\partial P/\partial\nu-k^2T)]^{\frac{1}{2}}, \tag{23}$$

which is precisely the exponent for Taylor instability with surface tension taken into account (Bellman & Pennington 1954). The disturbance amplitude will grow or decay with time depending on whether

$$\partial P/\partial\nu > k^2T^2 \quad\text{or}\quad \partial P/\partial\nu < k^2T, \tag{24}$$

respectively. Equation (24) is equivalent to the Taylor instability criterion adopted in our previous work (Dagan & Tulin 1972) if surface tension is neglected.

If $\partial P/\partial\nu=-1$, i.e. the pressure distribution is hydrostatic, μ becomes

$$\mu = \mu_w = i[k(1+k^2T)]^{\frac{1}{2}}, \tag{25}$$

which leads via (21) to the well-known dispersion relationship for progressive gravity waves in the presence of surface tension ($k^2T=\lambda'^2/\lambda_0'^2$, where

$$\lambda_0' = 2\pi(T'/\rho'g')^{\frac{1}{2}} = 1{\cdot}71\,\text{cm}$$

for air and water at 20 °C).

(ii) If $\partial P/\partial\nu-k^2T=0$, i.e. the net normal pressure gradient vanishes or the flow is free of gravity and surface tension, (22) yields

$$\mu = G. \tag{26}$$

Equations (21) and (15) lead in this case to

$$D(a_1\,\delta S)/Dt = 0, \tag{27}$$

where $a_1=\eta_1(0)\,e^{\mu t}$ is the disturbance amplitude. Equation (27) is just a wave continuity equation which is a direct result of the kinematical equation (18). It states that, as the wave is convected with the local velocity $\partial\phi/\partial\beta$, its amplitude grows or decays depending on whether $G>0$ or $G<0$.

Now we are in a position to draw a few conclusions on the local growth or decay of a disturbance in the general case in which the effect of the flow non-uniformity, expressed by G, and that of the normal acceleration, related to $\partial P/\partial\nu-k^2T$, are of the same order of magnitude. Inspection of the real part of μ in (22) leads to the following results.

(i) $\partial P/\partial \nu - k^2 T > 0$ (Taylor instability criterion) is a sufficient condition for the local growth of a disturbance. The rate of growth, however, is different from μ_T, given in (23), viz. $\mu < \mu_T$ for an accelerated flow and $\mu > \mu_T$ for a contractive motion.

(ii) $G > 0$ (contractive flow) is a sufficient condition for the local growth of the disturbance. However, if $\partial P/\partial \nu < k^2 T - G^2/4k$, the rate of growth is $\frac{1}{2}G$ and the disturbing wave is dispersive, and if $\partial P/\partial \nu > k^2 T - (G^2/4k)$, the wave is not dispersive and $\mu > \frac{1}{2}G$.

(iii) If $G < 0$ (stretching flow) the amplitude decreases unless $\partial P/\partial \nu > k^2 T$.

5. Global stability of the free-surface flow

We may relate the stability of the flow, in a global sense, to the breaking of the disturbing wave when its steepness exceeds a certain limiting value. Our local analysis is of limited value in this sense because of two basic limitations: (i) as the disturbance travels through regions of varying G and $\partial P/\partial \nu$, the local analysis has to be replaced by solving the complete set (8)–(10) and (ii) if the amplitude grows beyond a certain limit the analysis has to be supplemented by taking into account nonlinear terms in ϵ (e.g. Rajappa 1970).

The local analysis reveals, nevertheless, a few distinctive features of non-uniform flow as compared with the simple case of a plane free surface: as the disturbance travels along the free surface its rate of growth varies, as well as its relative phase velocity; in particular the disturbance may move from a region of growth to one of decay and the flow may be globally stable, at least for a certain range of initial disturbance amplitudes and wavelengths, although locally there is growth.

Solving the system (8)–(10) for some simple basic flows is probably the next step towards a better quantitative description of the global stability. We shall consider, nevertheless, a few examples in which the simplified local analysis may be still illuminating.

6. Discussion of a few examples

We are going to analyse two cases: in the first one the Taylor instability mechanism dominates, and in the second one the non-uniformity of the flow has the most important effect.

The rising bubble†

We consider the flow around a gaseous bubble, of constant gas pressure, which rises in a liquid of infinite extent. It is well known (Batchelor 1967, p. 475) that such bubbles are almost spherical (figure 3) and move at a constant speed

† I am indebted to Prof. Brooke Benjamin for drawing my attention to this example in a private discussion.

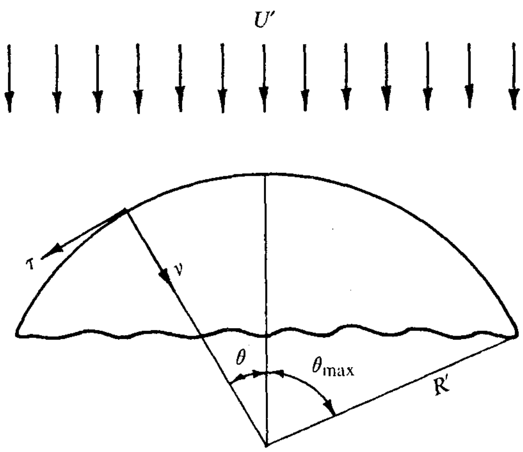

FIGURE 3. A rising gaseous bubble.

$U' \simeq \frac{2}{3}(g'R')^{\frac{1}{2}}$. With R' as reference length we may write (figure 3), by using Bernoulli's equation and the sphericity of the shape,

$$V = [2(1-\cos\theta)]^{\frac{1}{2}} = 2\sin\tfrac{1}{2}\theta, \qquad (28)$$

$$\partial P/\partial \nu = \cos\theta - V^2 = 3\cos\theta - 2.$$

We are going to examine the variation of μ along the free surface. If we assume that the free-surface disturbance is a spherical wave propagating from the stagnation region towards the outskirts, we have

$$G = -\frac{1}{\delta S}\frac{D(\delta S)}{Dt} = -\frac{\partial V}{\partial \theta} + \frac{V}{tg\theta} = -\left(\cos\tfrac{1}{2}\theta + \frac{\cos\theta}{\cos\frac{1}{2}\theta}\right). \qquad (29)$$

We have, in this case, an accelerated flow with the gravitational acceleration acting outwards from the liquid. At the stagnation point

$$\partial P/\partial \nu - k^2T = 1 - \lambda'^2/\lambda_0'^2 \quad (\theta = 0). \qquad (30)$$

Hence, if the wavelength of the disturbance is smaller than λ_0', μ is negative and any disturbance will decay. If $\lambda' > \lambda_0'$ (the Taylor criterion) μ is positive and the amplitude grows. For an observer moving with the disturbance the rate of change of the amplitude is

$$\frac{1}{a_1}\frac{\partial a_1}{\partial \theta} = \frac{1}{V}\frac{1}{a_1}\frac{Da_1}{Dt} = \frac{\mu(\theta)}{V(\theta)}. \qquad (31)$$

The ratio μ/V has its maximum at the stagnation point (where it tends to infinity) and decreases with θ, vanishing at an angle θ_0 given by

$$\theta_0 = \arccos\tfrac{1}{3}[2 + (\lambda_0'/\lambda')^2]. \qquad (32)$$

θ_0, which separates the regions of growth and decay, increases rapidly with λ'/λ_0' to its limiting value $\theta_{0\,\max} \simeq 48°$. This value is suprisingly close to the angle of the edge of the spherical caps found in experiments, $\theta_{\max} = 46°–64°$ (Batchelor 1967, p. 476).

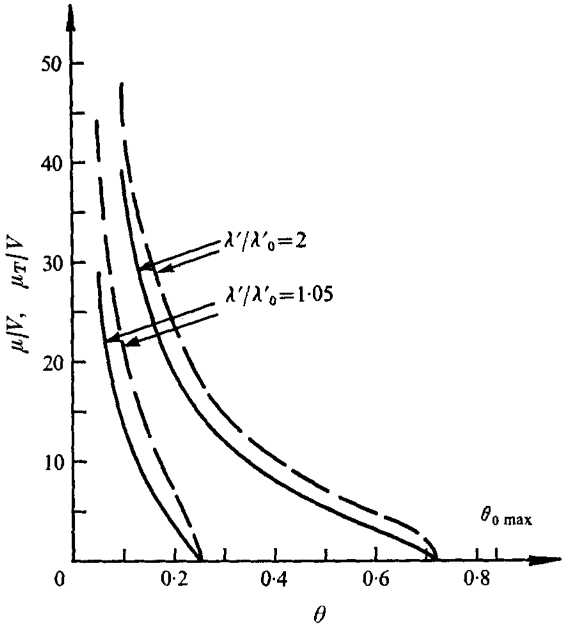

FIGURE 4. Variation of μ/V [from (22), (28) and (29)] and μ_T/V [from (23), (28) and (29)] along the bubble profile. $\lambda'_0/R' = 0{\cdot}100$. ——, μ/V; – – –, μ_T/V.

In figure 4 we have represented the variation of μ/V [from (22), (28), and (29)] with θ, and also of μ_T/V [from (23)], for $\lambda'_0/R' = 0{\cdot}1$ and the two values

$$\lambda'/\lambda'_0 = 1{\cdot}05 \quad \text{and} \quad 2,$$

in the range $\lambda'/2R' < \theta < \theta_0$. It is seen that application of the Taylor criterion, with neglect of the rate of strain G, leads to an overestimate of the rate of growth of the amplitude, which is particularly significant for λ' close to λ'_0.

We may draw the following conclusions on a tentative basis: (i) small bubbles ($R' \sim \lambda'_0$) are unconditionally stable because of the damping effect of the surface tension; (ii) for large bubbles ($R' > \lambda'_0$) small disturbances ($\lambda'_0 < \lambda' < R'$) propagate from the tip and reach their maximum amplitude at $\theta \simeq 48^\circ$, which is the probable region of collapse; (iii) for very large bubbles the instability at the tip is governed by the Taylor criterion and instability may occur near the stagnation point. Such a phenomenon has been observed for two-dimensional bubbles by Rowe & Partridge (1964) and it leads to the partition of the bubbles into twin smaller bubbles.

Stokes waves

The second example is that of progressive gravity waves in water. Two such waves, of different amplitude, are represented in figure 5. For such waves the Taylor instability criterion is never satisfied since, as is well known (Wehausen & Laitone 1960), $\partial P/\partial \nu \leqslant -0{\cdot}5$, the limiting value being attained at the crest of a wave of maximum limiting height. Hence, in most cases $\operatorname{Re}\mu = \frac{1}{2}G$ and the amplitude of a small disturbing wave grows as it travels from the trough towards the

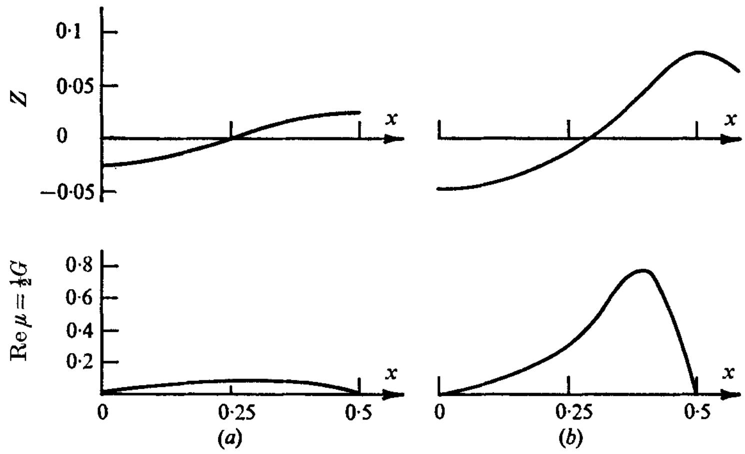

FIGURE 5. Variation of μ [from (22)] and profiles of water waves. (*a*) First-order Stokes wave in deep water ($2\epsilon_0 = 0{\cdot}05$). (*b*) Wave of finite amplitude in water of finite depth (Von Schwind & Reid 1972, $2\epsilon_0 = 0{\cdot}129$).

crest, in the region of contractive motion, and decreases afterwards. The disturbance wave is dispersive, the phase velocity relative to the main wave being related to $\mathrm{Im}\,\mu$.

To obtain a picture of the magnitude of $\frac{1}{2}G$ in the region of growth we have considered two examples, which differ in amplitude.

(*a*) *A first-order deep-water Stokes wave.* With L' equal to the wavelength we have

$$\left.\begin{aligned} Z &= -\epsilon_0 \cos 2\pi x, \quad \Phi = x - \epsilon_0 e^{2\pi y} \sin 2\pi x, \\ \mathrm{Re}\,\mu &= \tfrac{1}{2}G = \pi\epsilon_0 \sin 2\pi x(1 + 2\pi\epsilon_0 \cos 2\pi x) + O(\epsilon_0^3), \end{aligned}\right\} \tag{33}$$

where ϵ_0 is half the amplitude/wavelength ratio. The wave profile and $\mathrm{Re}\,\mu$ have been represented in figure 5(*a*) for $\epsilon_0 = 0{\cdot}025$. A disturbance travelling from the trough attains its maximum amplitude at the crest. Since G is a slowly varying function of x we have approximately

$$\frac{1}{a_1}\frac{\partial a_1}{\partial s} = \frac{G}{2V} = \frac{\epsilon_0 \sin 2\pi x}{2} + O(\epsilon_0^2) \tag{34}$$

and

$$\frac{a_{1\,\mathrm{crest}}}{a_{1\,\mathrm{trough}}} = e^{\epsilon_0/2\pi} + O(\epsilon_0^2) = 1 + \frac{\epsilon_0}{2\pi} + O(\epsilon_0^2). \tag{35}$$

The ratio (35) is precisely the result obtained by Longuet-Higgins & Stewart (1960), as it should be in the case of a basic flow which differs slightly from a uniform flow.

(*b*) *A wave of finite amplitude* (figure 5*b*). We have adopted in this case the solution of Von Schwind & Reid (1972), who have determined the potential and the free-surface profile by a ten-term Fourier series† The wave represented in figure 5(*b*) corresponds to $2\epsilon_0 \simeq 0{\cdot}129, h'/L' = 0{\cdot}409$ and $C'/(gL')^{\frac{1}{2}} = 0{\cdot}42$, where

† We were not aware of the work of Schwartz (1974) when this calculation was done.

h' is the water depth and C' is the Rayleigh wave speed. This is the wave of maximum height, for the given h'/L', which could be determined before the computer results began to diverge. Although the accuracy of the solution is questionable at such a large amplitude, it still permits us to obtain a qualitative picture of the phenomenon.

First it should be mentioned that the square-root term in μ in (22) is imaginary, since only for $k < 0{\cdot}65$ does the expression $G^2_{\text{max}} - 4k$ become positive. Our local analysis does not apply, however, to disturbances with such a low wavenumber.

The distribution of $\text{Re}\,\mu = \frac{1}{2}G$ as function of x is represented in figure 5(b); it differs markedly from that obtained by the linear analysis (figure 5a), because of the nonlinear effects, which cause an increase of $\sin\theta$ and decrease of V at the point of G_{max}, for instance.

To conclude, the growth of the amplitude of a small disturbance is governed in the case of gravity waves by the rate of strain on the free surface. The rate of growth increases nonlinearly with the amplitude of the gravity wave and probably becomes very large when the wave approaches its limiting height. The cumulative effect of amplitude growth reaches its maximum at the crest and breaking will start, therefore, in its neighbourhood.

7. Conclusions

The present study has shown that the growth or decay of a small disturbance in a free-surface non-uniform flow is governed by two mechanisms: the kinematical effect of the rate of strain on the free surface and the dynamical effect of the normal pressure gradient, the latter being related to the Taylor instability. A local analysis, valid for disturbances of large wavenumbers, has led to a simple expression for the rate of growth of the amplitude.

The problem of the stability of non-uniform free-surface flows is complex. If we relate instability to the breaking of the disturbing wave, a complete analysis has to take into account nonlinear effects related to the amplitude and to the interaction with the basic flow, as the disturbance travels through regions of growth or decay. Moreover, viscosity will probably play a role in the development of small disturbances. The simplified analysis presented here still reveals some qualitative features of the mechanism of stability of free-surface flows.

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